ON CERTAIN PROBABILITY DISTRIBUTIONS ARISING FROM A SEQUENCE OF OBSERVATIONS AND THEIR APPLICATIONS

BY P. V. KRISHNA IYER
Defence Science Laboratory

AND

B. N. SINGH
*Banaras Hindu University**

INTRODUCTION

THE distributions of a number of *statistics* defined for a sequence of n observations $x_1, x_2 \ldots x_n$ taking any of the values $\theta_1, \theta_2, \theta_3 \ldots \theta_k$ with fixed or varying probabilities have been considered by Krishna Iyer (1948–54), Mood (1940) and others. These distributions refer mainly to *statistics* obtained by considering the relations between adjoining observations as in the case of a simple Markoff chain. For a binomial sequence, Singh (1952) has discussed some distributions based on the relationship between three adjacent observations. Similar distributions of a wider nature have been discussed by Kendall (1945), Wilcoxon (1945), Mann and Whitney (1947), Rijkoort (1952), Kruskall (1952), Mood (1940), Stuart (1955) and others. Kendall's (1945) rank correlation τ is based on $(x - y)$ where x and y are the number of positive and negative differences between any two pairs of observations for a random sequence drawn from a continuous distribution. For two random samples x and y from a continuous distribution $F(x)$, Mann and Whitney (1947) have considered the U-*statistic*. This *statistic* is defined as the number of times that the y's precede the x's when the two samples x and y are pooled together and arranged in ascending or descending order. Another *statistic* T, closely related to U, was given by Wilcoxon (1945) earlier, where T represents the sum of the ranks of y's when the two samples taken together are arranged in ascending order. It has been shown by Mann and Whitney that

$$U = mn + \frac{m(m+1)}{2} - T,$$

---

* Now in the Indian Standards Institution.

where n and m are the sizes of the samples x and y. Whitney (1951) has extended the U-*statistic* for three samples, x, y and z by introducing the *statistics* U and V, where U and V represent the number of times that y and z precede those of x when the three samples are pooled and arranged in ascending order. Rijkoort (1952) generalized Wilcoxon's test to k samples by taking the *statistic*

$$s = \Sigma (s_i - n_i\bar{r})^2$$

where $\bar{r} = \frac{1}{2}(\Sigma n_i + 1) = \frac{1}{2}(n + 1)$, s_i is equal to the sum of the ranks of x_i; and n_i is the size of the sample x_i. Kruskall and Wallis (1952) also have considered similar *statistics*. Mood (1954) has used a similar method for testing the difference in the dispersion of two samples x and y. This test depends on

$$W = \sum_{i=1}^{n} \left(r_i - \frac{m + n + 1}{2}\right)^2,$$

where r_i is the rank of the ith observation in y when x and y are arranged together in order of magnitude.

It would be seen that the work done so far for a sequence of observations related mainly to distributions based on the relations between either adjoining pairs or all possible pairs of observations from a continuous population. The purpose of this paper is to investigate the possibility of developing non-parametric tests more powerful than the existing ones by studying the distributions of a number of new *statistics* arising from a sequence of observations from a continuous or discrete population by taking the differences between all pairs separated by r or less number of observations. The value of these *statistics* for testing the randomness of a sequence of observations or for examining whether two or more samples belong to the same parent population has been investigated by working out the power and the efficiency of the various tests arising from this investigation.

2. Differences between Three Successive Observations

A. *Positive or negative differences*

A given sequence of n observations can be considered as $(n - 2)$ sets of three successive values. Each of these sets gives three differences which are either positive, negative or zero. By considering the number of positive or negative differences in the $(n - 2)$ sets we shall define two statistics, W_3 and T_3 as follows:—

$$W_3 = X_1 + X_2$$

$$T_3 = X_1' + X_2$$

where

$$X_1 = (12) + 2\,(23) + 2\,(34) + \dots + 2\,(n-2, n-1) + (n-1, n)$$

$$X_1' = (12) + (23) + (34) + \dots + (n-2, n-1) + (n-1, n)$$

$$X_2 = (13) + (24) + (35) + \dots + (n-2, n)$$

and (rs) represents the sign of the difference between the r-th and the s-th observations of the sequence and is assigned the scores 1 or 0 according as $(x_r - x_s)$ is positive or otherwise when the distribution considered is that for positive differences. While considering the distribution for negative differences the scores assigned to $(x_r - x_s)$ are -1 or 0 according as $(x_r - x_s)$ is negative or otherwise. It may be noted that W_3 represents the total number of positive or negative differences arising from the $(n-2)$ moving sets or blocks of 3 consecutive observations. The differences considered in the s-th set are those between the observations $(s, s+1)$, $(s+1, s+2)$ and $(s, s+2)$. T_3 represents the total number of positive or negative differences between any two observations r and s such that $s - r \leqslant 2$. The probability and the moment generating functions (P.G.F. and M.G.F.) for the distributions of W_3 and T_3 obtained by the methods developed by Iyer (1950) are given below:—

(*a*) *P.G.F. and cumulants of* W_3 *for two and three characters.*—

Assuming $\phi(n)$ to be the P.G.F. of W_3 for n observations which take the values θ_1 and θ_2 with fixed probabilities p and q, the following recurrence relationship holds good for this distribution:—

$$\phi(n+3) - \phi(n+2) + pq(1-\xi^2)\,\phi(n+1) + pq\xi^2(1-\xi)\,\phi(n) - p^2q^2\xi^2(1-\xi)^2\,\phi(n-1) = 0 \quad (2.1)$$

where

$$\phi(n) = p(n,0) + \xi\,p(n,1) + \xi^2\,p(n,2) + \dots,$$

and $p(n, r)$ is the probability of getting r positive or negative differences for W_3 from n observations. The distribution of W_3 for $n \geqslant 5$ can be obtained in succession from those of the lower values, *viz.*, $n = 4, 3, 2$ which are actually determined by examining the different possible arrangements.

The difference equation for the M.G.F. is the same as (2.1) with ξ replaced by e^t. Thus the M.G.F. for W_3 is

$$M(n+3) - M(n+2) + pq(1-e^{2t}) M(n+1) + pqe^{2t}(1-e^{t}) M(n) - p^2q^2e^{2t}(1-e^{t})^2 M(n-1) = 0 \tag{2.2}$$

where

$$M(n) = 1 + t\mu_1'(n) + \frac{t^2}{2!}\mu_2'(n) + \frac{t^3}{3!}\mu_3'(n) + \ldots$$

The solution of (2.2) is given by

$$M(n) = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n + c_4\alpha_4^n \tag{2.3}$$

where α_1, α_2, α_3, α_4 are the roots of the characteristic equation of the recurrence relation (2.2), *viz.*,

$$x^4 - x^3 + pq(1-e^{2t}) + pqe^{2t}(1-e^{t}) - p^2q^2e^{2t}(1-e^{t})^2 = 0 \tag{2.4}$$

and the c's are constants determined by equating (2.3) to the actual M.G.F.'s for $n = 4$, 3, 2 and 1.

When $t = 0$ (2.4) has all the roots, excepting one, equal to zero. If this non-zero root is α_1, then

$$M(n) = c_1\alpha_1^n \left\{1 + \frac{c_2}{c_1}\left(\frac{\alpha_2}{\alpha_1}\right)^n + \frac{c_3}{c_1}\left(\frac{\alpha_3}{\alpha_1}\right)^n + \frac{c_4}{c_1}\left(\frac{\alpha_4}{\alpha_1}\right)^n\right\}$$

$$= c_1\alpha_1^n\beta \text{ say.}$$

Taking the logarithm of $M(n)$, we get the cumulants of W_3.

The r-th cumulant, κ_r, is equal to

$$\left[\frac{d^r}{dt^r}\log M_n\right]_{t=0} = \left[\frac{d^r}{dt^r}\log c_1\right]_{t=0} + n\left[\frac{d^r}{dt^r}\log \alpha_1\right]_{t=0} + \left[\frac{d^r}{dt^r}\log\beta\right]_{t=0} \tag{2.5}$$

It can be easily seen that

$$\left[\frac{d^r}{dt^r}\log\beta\right]_{t=0} = 0$$

so long as $r < n$, as α_2, α_3 and α_4 are zero when $t = 0$. Therefore (2.5) reduces to

$$\kappa_r = \left[\frac{d^r}{dt^r}\log c_1\right]_{t=0} + n\left[\frac{d^r}{dt^r}\log \alpha_1\right]_{t=0}$$

When n is large, the contribution of c_1 will be negligible compared to n and therefore

$$\kappa_r \sim n \left[\frac{d^r}{dt^r} \log \alpha_1\right]_{t=0}$$

Now

$$\left[\frac{d^r}{dt^r} \log \alpha_1\right]_{t=0}$$

can be obtained by differentiating the characteristic equation of the M.G.F. r times with respect to t as has been indicated in a previous publication by Krishna Iyer and Kapur (1955). This aspect is being discussed in greater detail in another paper to be published shortly in this journal. It may also be noted that by taking $(d^r/d\xi^r \log \alpha_1)$ we get the factorial cumulants $\kappa_{[r]}$ and the relation between the factorial cumulants and the ordinary cumulants is given by

$$\kappa_r = \kappa_{[r]} + \kappa_{[r-1]}\, \Delta O^r + \kappa_{[r-2]}\, \Delta^2 O^r + \ldots + \kappa_{[r-s]}\, \Delta^s O^r + \ldots$$

where $\Delta^s O^r$ is the s-th difference of O.

For three characters the recurrence relationship for the P.G.F. of W_3 is

$$\begin{aligned}
[E^9 - E^8 + E^7 (1 - \xi^2) \Sigma p_i p_j + E^6 (1 - \xi) \{\xi^2 \Sigma p_i p_j \\
- p_1 p_2 p_3 (1 + 2\xi + 3\xi^2 + \xi^3)\, (1 - \xi)\} - E^5 \xi^2 (1 - \xi)^2 \\
\times \{\Sigma p_i^2 p_j^2 + p_1 p_2 p_3 (2 + 2\xi + \xi^2)\} - E^4 \xi^2 (1 - \xi)^2 p_1 p_2 p_3 \\
\times \{\xi^2 - (1 - \xi^3) \Sigma p_i p_j\} + E^3 \xi^4 (1 - \xi)^3 p_1 p_2 p_3 \{\Sigma p_i p_j \\
+ \xi (1 - \xi) p_1 p_2 p_3\} - E^2 \xi^4 (1 - \xi)^4 (1 + \xi^2) p_1^2 p_2^2 p_3^2 \\
- E \xi^6 (1 - \xi)^5 p_1^2 p_2^2 p_3^2 \Sigma p_i p_j \\
+ \xi^6 (1 - \xi)^6 p_1^3 p_2^3 p_3^3]\, \phi (n - 1) = 0
\end{aligned} \qquad (2.6)$$

where E stands for the usual operator defined by

$$E \phi (n) = \phi (n + 1),$$

$$\Sigma p_i p_j = p_1 p_2 + p_1 p_3 + p_2 p_3$$

and

$$\Sigma p_i^2 p_j^2 = p_1^2 p_2^2 + p_2^2 p_3^2 + p_1^2 p_3^2$$

The first four asymptotic cumulants for two characters calculated by the method given above are noted below:—

$$\left.\begin{aligned}
&\kappa_1 = 3npq; \quad \kappa_2 = npq\,(9 - 31pq) \\
&\kappa_3 = npq\,(27 - 297pq + 732p^2q^2) \\
&\kappa_4 = npq\,(81 - 2197pq + 14376p^2q^2 - 27474p^3q^3)
\end{aligned}\right\} \qquad (2.7)$$

The exact expressions for κ_1, κ_2 and κ_3 for any number of characters for free sampling (*i.e.*, fixed probabilities) are as follows:—

$$\left.\begin{aligned} \kappa_1 &= 3(n-2)\,a_2 \\ \kappa_2 &= (9n-22)a_2 + (22n-70)\,a_3 - (31n-92)\,a_2^2 \\ \kappa_3 &= 6(15n-52)\,A + 54(3n-14)\,B \\ &\quad + 3(3n-8)\,C + 6(3n-8)\,D \\ &\quad + 24(n-3)\,F + 12(3n-11)\,G \\ &\quad + 48(n-4)\,H + 12(n-3)\,N \end{aligned}\right\} \quad (2.8)$$

where

$$\begin{aligned} a_2 &= \Sigma p_i p_j, \quad a_3 = \Sigma p_i p_j p_k, \quad a_4 = \Sigma p_i p_j p_k p_l, \\ A &= (1-2a_2)(a_3 - a_2^2), \quad B = a_4 - 2a_3 a_2 + a_2^3 \\ C &= a_2(1-a_2)(1-2a_2), \quad D = (1-2a_2)(a_2 + a_3 - 2a_2^2) \\ F &= a_3 - a_2(a_2 + 2a_3) + 2a_2^3 \\ G &= (\Sigma p_i^2 p_j p_k + \Sigma p_i p_j p_k^2 + 4a_4) - a_2(\Sigma p_i^2 p_j + \Sigma p_i p_j^2 \\ &\qquad + 8a_3) + 4a_2^3 \\ H &= (\Sigma p_i^2 p_j p_k + 2\Sigma p_i p_j^2 p_k + \Sigma p_i p_j p_k^2 + 6a_4) \\ &\qquad - a_2(\Sigma p_i^2 p_j + \Sigma p_i p_j^2 + 6a_3) + 2a_2^3 \\ N &= (\Sigma p_i^2 p_j^2 + 2\Sigma p_i^2 p_j p_k + \Sigma p_i p_j^2 p_k + 2\Sigma p_i p_j p_k^2 + 5a_4) \\ &\qquad - a_2(\Sigma p_i^2 p_j + \Sigma p_i p_j^2 + 4a_3) + a_2^3 \end{aligned}$$

By putting $a_2 = \frac{1}{2}$ and $a_3 = 1/6$ in the above expressions, we obtain the cumulants of W_3 for a sequence of observations from a continuous population.

For non-free sampling, *i.e.*, when the number of observations taking the values of $\theta_1, \theta_2 \ldots . \theta_k$ is $n_1, n_2, \ldots . n_k$ respectively such that $\Sigma n_i = n$, κ_1 and κ_2 reduce to

$$\left.\begin{aligned} \kappa_1 &= 3(n-2)\,\frac{\Sigma n_i n_j}{n(n-1)} \\ \kappa_2 &= (9n-22)\,\frac{\Sigma n_i n_j}{n(n-1)} \\ &\quad + (9n^2-67n-128)\,\frac{\Sigma n_i(n_i-1)\,n_j(n_j-1)}{n(n-1)(n-2)(n-3)} \\ &\quad + 2(9n^2-56n+93)\,\frac{\Sigma n_i n_j n_k}{n(n-1)(n-2)} \\ &\quad - 2(9n^2-67n+128)\,\frac{\Sigma n_i n_j n_k n_l}{n(n-1)(n-2)(n-3)} \\ &\quad - \left[3(n-2)\,\frac{\Sigma n_i n_j}{n(n-1)}\right]^2 \end{aligned}\right\} \quad (2.9)$$

The above values have been obtained from the uncorrected moments about the origin zero by substituting

$$\frac{n_i^{[r]} n_j^{[s]} n_k^{[t]} \ldots}{n^{[r+s+t+\cdots]}} \quad \text{for } p_i^r p_j^s p_k^t \ldots \tag{2.10}$$

in the moments about the origin *zero* for free sampling.

(*b*) *P.G.F. and cumulants for* T_3.—The recurrence relation for the P.G.F. of the distribution of T_3 for two characters reduces to

$$\phi(n+3) - \phi(n+2) + pq(1-\xi)\phi(n+1)$$
$$+ pq\xi(1-\xi)\phi(n) - p^2q^2\xi(1-\xi)^2\phi(n-1) = 0 \quad (2.11)$$

The asymptotic values of the first four cumulants are

$$\left.\begin{aligned} \kappa_1 &= 2npq, \ \kappa_2 = 2npq(2 - 7pq) \\ \kappa_3 &= 2npq(4 - 45pq + 113p^2q^2) \\ \kappa_4 &= 2npq(8 - 223pq + 1554p^2q^2 - 2910p^3q^3) \end{aligned}\right\} \quad (2.12)$$

The actual values of κ_1 and κ_2 for k characters or variables are as under:—

$$\left.\begin{aligned} \kappa_1 &= (2n-3)\,a_2 \\ \kappa_2 &= (4n-7)\,a_2 + 2(5n-14)\,a_3 - 7(2n-5)\,a_2^2 \end{aligned}\right\} \quad (2.13)$$

For non-free sampling κ_1 and κ_2 reduce to

$$\left.\begin{aligned} \kappa_1 &= (2n-3)\frac{\Sigma n_i n_j}{n(n-1)} \\ \kappa_2 &= (4n-7)\frac{\Sigma n_i n_j}{n(n-1)} \\ &\quad + (4n^2 - 26n + 44)\frac{\Sigma n_i^{[2]} n_j^{[2]}}{n^{[4]}} \\ &\quad + 2(4n^2 - 21n + 30)\frac{\Sigma n_i n_j n_k}{n^{[3]}} \\ &\quad - 2(4n^2 - 26n + 44)\frac{\Sigma n_i n_j n_k n_l}{n^{[4]}} \\ &\quad - \left[(2n-3)\frac{\Sigma n_i n_j}{n(n-1)}\right]^2 \end{aligned}\right\} \quad (2.14)$$

B. *Positive and negative differences*

Assuming W_3' and T_3' to be the *statistics* corresponding to W_3 and T_3 obtained by taking positive and negative differences from three successive observations, the P.G.F.'s and the cumulants of the two distributions for two or more characters are noted below:—

P.G.F. and cumulants of W_3' *for two and three characters.*—For two characters the recurrence relationship reduces to

$$\phi(n+3)-\phi(n+2)+pq(1-\xi^4)\,\phi(n+1)+pq\xi^4(1-\xi^2)\,\phi(n) - p^2q^2\xi^4(1-\xi^2)^2\,\phi(n-1)=0 \qquad (2.15)$$

For three characters the recurrence relation is given by a 9×9 determinant which on expansion reduces to

$$\begin{aligned}
[E^9 &- E^8 + E^7(1-\xi^4)\,a_2 + E^6\{\xi^4(1-\xi^2)\,a_2 - (1-3\xi^6 \\
&+ 2\xi^9)\,p_1p_2p_3\} - E^5.\xi^4(1-\xi)\{(1-\xi^2)(1+\xi)\,\Sigma p_i^2p_j^2 \\
&+ (2+2\xi-\xi^4-3\xi^5)\,p_1p_2p_3\} - E^4\,\xi^4(1-\xi)^2\,p_1p_2p_3 \\
&\times \{\xi^4(1+2\xi) - (1+\xi)(1+\xi-2\xi^5)\,a_2\} + E^3.\xi^8 \\
&\times (1-\xi)^3\,p_1p_2p_3\{(1+\xi)(1+2\xi)\,a_2 + \xi(2+3\xi+3\xi^2 \\
&- \xi^3(1+\xi)^3\,p_1p_2p_3\} \\
&- E^2\xi^8(1+2\xi)(1-\xi)^2(1-\xi^2)^2(1-\xi^4)\,p_1^2p_2^2p_3^2 \\
&- E\xi^{12}(1-\xi)^4(1-\xi^2)(1+2\xi)^2\,p_1^2p_2^2p_3^2a_2 \\
&+ \xi^{12}(1-\xi)^6(1+2\xi)^3p_1^3p_2^3p_3^3]\,\phi(n-1) = 0 \qquad (2.16)
\end{aligned}$$

where

$$a_2 = p_1p_2 + p_1p_3 + p_2p_3$$

The asymptotic values of the first four cumulants for two characters are as follows:—

$$\left.\begin{aligned}
\kappa_1 &= 6npq,\ \kappa_2 = 4npq\,(9-31\,pq) \\
\kappa_3 &= 8npq\,(27-302pq+732p^2q^2) \\
\kappa_4 &= 16npq\,(81-2707pq+16692p^2q^2-31986p^3q^3)
\end{aligned}\right\} \qquad (2\cdot 17)$$

The actual values of the first and second cumulants for any number of variables or characters are given below for infinite or free sampling:

$$\left.\begin{aligned}
\kappa_1 &= 6(n-2)\,a_2 \\
\kappa_2 &= 4(9n-26)\,a_2 + 6(13n-40)\,a_3 - 4(31n-92)\,a_2^2
\end{aligned}\right\} \qquad (2\cdot 18)$$

For finite sampling the above formulæ reduce to

$$\left.\begin{aligned}
\kappa_1 &= 6\,(n-2)\,\frac{\Sigma\, n_i n_j}{n\,(n-1)}\\
\kappa_2 &= 4\,(9n-26)\,\frac{\Sigma\, n_i n_j}{n\,(n-1)} + 2\,(36n^2-229n\\
&\quad + 392)\,\frac{\Sigma\, n_i n_j n_k}{n^{[3]}}\\
&\quad + (36n^2-268n+512)\,\frac{\Sigma\, n_i^{[2]} n_j^{[2]}}{n^{[4]}}\\
&\quad + 2\,(36n^2-268n+512)\,\frac{\Sigma\, n_i n_j n_k n_l}{n^{[4]}}\\
&\quad - \left[6\,(n-2)\,\frac{\Sigma\, n_i n_j}{n\,(n-1)}\right]^2
\end{aligned}\right\} \quad (2{\cdot}19)$$

P.G.F. and cumulants of T_3'

The recurrence relation for two characters for T_3 is given by

$$\phi\,(n+3) - \phi\,(n+2) + pq\,(1-\xi^2)\,\phi\,(n+1)$$
$$+ pq\xi^2\,(1-\xi^2)\,\phi\,(n) - p^2q^2\xi^2\,(1-\xi^2)^2\,\phi\,(n-1) = 0 \quad (2{\cdot}20)$$

The first four asymptotic values of the cumulants for two characters reduce to

$$\left.\begin{aligned}
\kappa_1 &= 4npq,\ \ \kappa_2 = 8npq\,(2-7pq)\\
\kappa_3 &= 16npq\,(4-45pq+113p^2q^2)\\
\kappa_4 &= 32npq\,(8-223pq+1494p^2q^2-2910p^3q^3)
\end{aligned}\right\} \quad (2{\cdot}21)$$

The actual first and second cumulants of T_3' for free sampling for k characters are

$$\left.\begin{aligned}
\kappa_1 &= 2\,(2n-3)\,a_2\\
\kappa_2 &= 2\,(8n-19)\,a_2 + 12\,(3n-8)\,a_3 - 28\,(2n-5)\,a_2^2
\end{aligned}\right\} \quad (2{\cdot}22)$$

For finite sampling κ_1 and κ_2 work out to

$$\left.\begin{aligned}
\kappa_1 &= 2\,(2n-3)\,\frac{\Sigma\, n_i n_j}{n\,(n-1)}\\
\kappa_2 &= 2\,(8n-19)\,\frac{\Sigma\, n_i n_j}{n\,(n-1)}\\
&\quad + 2\,(16n^2-86n+128)\,\frac{\Sigma\, n_i n_j n_k}{n^{[3]}}\\
&\quad + (16n^2-104n+176)\,\frac{\Sigma\, n_i^{[2]} n_j^{[2]}}{n^{[4]}}\\
&\quad - 2\,(16n^2-104n+176)\,\frac{\Sigma\, n_i n_j n_k n_l}{n^{[4]}}\\
&\quad - \left\{2\,(2n-3)\,\frac{\Sigma\, n_i n_j}{n\,(n-1)}\right\}^2
\end{aligned}\right\} \quad (2{\cdot}23)$$

3. Differences between r Successive Observations

In the previous section we considered the distribution of the number of positive or/and negative differences arising from three contiguous observations in a given sequence. We shall, now, investigate the distributions of W, W', T and T' for the general case of r consecutive observations. No general expressions, which will hold good for any value of r, exist for the variance and other higher cumulants of these distributions. In fact the results for the variances and higher cumulants for $r \leqslant (n/2 + 1)$ and $r > (n/2 + 1)$ differ and therefore we give the variances for the different distributions for these two cases separately. The exact probability generating functions and the recurrence relations satisfied by them for any value of r are rather complicated and, therefore, have not been discussed in this paper. We shall, however, discuss these distributions by obtaining their first and second cumulants and examining the nature of their higher order cumulants. It may be noted that for $r = n$, the distributions of the number of positive or negative signs for a continuous distribution is the same as that considered by Kendall (1945) in his discussions on rank, correlation coefficient τ.

A. *Positive or negative differences*

(*a*) *Statistics* W_r.—Let x_1, $x_2 \ldots x_n$ be a given sequence of observations taking any one of the values θ_1, θ_2, ... θ_k with probabilities p_1, p_2, $\cdots$ p_k subject to the condition $\sum_{i=1}^{k} p_i = 1$. Consider the signs of the differences (taken in the same sense or order) between all possible pairs of values arising from moving sets or blocks of r consecutive observations. The number of blocks that can be taken in such a scheme is $(n - r + 1)$. We shall now deal for the $(n - r + 1)$ blocks the distribution of the total for the number of positive or negative differences obtained by taking all possible differences from each of the $(n - r + 1)$ blocks of size r.

The distribution of the number of positive differences is evidently the same as that for negative differences. Taking $r \leqslant (n/2 + 1)$, let X_1, X_2, $\cdots$ X_{r-1} be defined by the relations

$$\left.\begin{aligned}
X_1 = {} & (12)+2\,(23)+3\,(34)+\cdots \\
& +(r-1)\,(r-1,\,r)+(r-1)\,(r,\,r+1) \\
& +\cdots+(r-1)\,(n-r+1,\,n-r+2) \\
& +\,(r-2)\,(n-r+2,\,n-r+3)+\cdots \\
& +\,3\,(n-3,\,n-2)+2\,(n-2,\,n-1) \\
& +\,(n-1,\,n) \\
X_2 = {} & (13)+2\,(24)+3\,(35)+\cdots \\
& +\,(r-2)\,(r-2,r)+(r-2)\,(r-1,\,r+1) \\
& +\cdots+(r-2)\,(n-r+1,\,n-r+3) \\
& +\,(r-3)\,(n-r+2,\,n-r+4)\,+\cdots \\
& +\,2\,(n-3,\,n-1)+(n-2,\,n) \\
X_3 = {} & (14)+2\,(25)+3\,(36)+\cdots \\
& +\,(r-3)\,(r-3,\,r)+(r-3)\,(r-2, \\
& r+1)+\cdots+(r-3)\,(n-r+1,\,n-r+4) \\
& +\,(r-4)\,(n-r+2,\,n-r+5)+\cdots \\
& +\,2\,(n-4,\,n-1)+(n-3,\,n) \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
X_{r-2} = {} & (1,\,r-1)+2\,(2,\,r)+2\,(3,\,r+1)+\ldots \\
& +\,2\,(n-r+1,\,n-1)+(n-r+2,\,n) \\
X_{r-1} = {} & (1,\,r)+(2,\,r+1)+(3,\,r+2)+\ldots \\
& +\,(n-r+1,\,n)
\end{aligned}\right\} \qquad (3\cdot1)$$

where $(i\,j)$ denotes the difference between the i-th and j-th observations and assumes the value 1 or 0 according as $(x_i - x_j)$ is positive or otherwise, if the distribution considered is that for positive signs. If the distribution considered is that for negative differences the scores given to $(x_i - x_j)$ are -1 and 0 according as $(x_i - x_j)$ is negative or otherwise.

The expectation for the total number (W_r) of positive signs in $(n - r + 1)$ moving blocks or sets, each consisting of r consecutive observations, is given by

$$E\,(W_r) = E\left(\sum_{h=1}^{r-1} X_h\right) = (n - r + 1)\binom{r}{2}\,a_2 \qquad (3.2)$$

where $a_2 = \Sigma\, p_i p_j$.

The variance of the distribution for $T = \Sigma X_h$ can be obtained by evaluating

$$E\left[\sum_{h=1}^{r-1}(X_h - \bar{X}_h)\right]^2 \tag{3.3}$$

Expanding (3.3) in terms of the substitutions given in (3.1) the variance reduces to

> k_1 {Variance of a positive difference like (12) from two observations} + $2k_2$ {covariance of two positive differences like (12) and (23) from three observations} + $2k_3$ {covariance of two positive differences like (12) and (13) from three observations} + $2k_4$ {covariance of two positive differences like (13) and (23) from three observations} or symbolically

$$k_1 \text{ var } (\underset{1\,2}{\widehat{x\,x}}) + 2k_2 \text{ cov } (\underset{1\;\;2\;3}{\widehat{x\,x}\widehat{\,x}}) + 2k_3 \text{ cov } (\underset{123}{\widehat{xxx}})$$
$$+ 2k_4 \text{ cov } (\underset{123}{\widehat{xxx}}) \tag{3.4}$$

where k_1, k_2, k_3, k_4 represent the number of times that the configurations associated with the respective k's would occur in the distribution. The variance and the covariances for these configurations are given in Table I.

TABLE I

Variance and covariances for different configurations

Configurations	Variance or Covariance	Remarks
	$a_2(1-a_2)$	$a_2 = \Sigma p_i p_j$
	$a_3 - a_2^2$	$a_3 = \Sigma p_i p_j p_k$
	$\Sigma p_i p_j^2 + 2a_3 - a_2^2$	..
	$\Sigma p_i^2 p_j + 2a_3 - a_2^2$	..

We shall now obtain the values of k_1, k_2, k_3 and k_4.

$$k_1 = \sum_{h=1}^{r-1}\left\{2\sum_{t=1}^{r-h-1} t^2 + (r-h)^2(n-2r+h+2)\right\}$$

$$= \frac{1}{12} r(r-1)\{2n(2r-1) - r(5r-7)\} \tag{3.5}$$

To determine k_2, we have to enumerate the number of ways in which any two observations, one on each side of the $(n-2)$ central values of the sequence, get associated with one another in the distribution under consideration. It can be seen that for the s-th observation, $s \leqslant (r-1)$, this number is equal to

$$\frac{s\,(s-1)}{2} \cdot \frac{s\,(2r-s-1)}{2} \tag{3.6}$$

The contribution in k_2 for $s \leqslant (r-1)$ is given by

$$\sum_{s=1}^{r-1} \frac{s\,(s-1)}{2} \cdot \frac{s\,(2r-s-1)}{2}$$

$$= \frac{1}{120}\, r\,(r-1)\,(r-2)\,(9r^2-8r+3) \tag{3.7}$$

On account of the symmetry, the contributions in k_2 for the first and the last $(r-1)$ observations are equal. The $(n-2r+2)$ observations in the centre will make a further contribution of

$$\frac{(n-2r+2)\,r^2\,(r-1)^2}{4} \tag{3.8}$$

to k_2. Hence

$$k_2 = \frac{1}{60}\, r\,(r-1)\,(r-2)\,(9r^2-8r+3)$$

$$+\tfrac{1}{4}\,(n-2r+2)\,r^2\,(r-1)^2 \tag{3.9}$$

The values of k_3 and k_4 are equal and k_3 can be obtained by noting the number of times that one observation gets associated with any two observations to its right in the distribution. The s-th observation $(s \leqslant r-1)$ can be associated with the remaining observations to the right of it in

$$\left\{\binom{\frac{s}{2}(2r-s-1)}{2} - (r-s)\binom{s}{2} - \binom{s-1}{2} - \binom{s-2}{2} - \cdots - \binom{2}{2}\right\} \tag{3.10}$$

ways. Now (3.10) reduces to

$$\left\{\binom{\frac{s}{2}(2r-s-1)}{2} - (r-s)\binom{s}{2} - \binom{s}{3}\right\} \tag{3.11}$$

The contribution in k_3 for $s \leqslant (r-1)$ and $> (n-r+1)$ is

$$\sum_{s=1}^{r-1}\left[\left\{\binom{\frac{s}{2}(2r-s-1)}{2} - (r-s)\binom{s}{2} - \binom{s}{3}\right\}\right.$$

$$\left.+\left\{\binom{\frac{s}{2}(s-1)}{2} - \binom{s}{3}\right\}\right] \qquad (3.12)$$

$$= \frac{1}{120}\, r\,(r-1)\,(r-2)\,(11r^2 - 17r + 12) \qquad (3.13)$$

For $(r-1) < s \leqslant (n-r+1)$, the contribution in k_3 is

$$(n-2r+2)\left\{\binom{\frac{r}{2}(r-1)}{2} - \binom{r}{3}\right\} \qquad (3.14)$$

The sum of the expression (3·13) and (3·14) is k_3.

Multiplying the k's by the respective variance and covariances of the configurations and simplifying, we get the variance or the second cumulant of W_r as

$$\kappa_2 = \frac{1}{60}\, r\,(r-1)\,[5\,\{2n\,(2r-1) - r\,(5r-7)\}\,(a_2 - a_2^2)$$
$$+ 2\,\{(r-2)\,(9r^2 - 8r + 3) + 15\,(n-2r+2)$$
$$\times\, r\,(r-1)\}\,(a_3 - a_2^2) + (r-2)\,\{(11r^2 - 17r + 12)$$
$$+ 5\,(n-2r+2)\,(3r-1)\}\,(a_2 + a_3 - 2a_2^2)] \qquad (3·15)$$

where the a's are monomial symmetric function in p's.

The general expressions for the mean and the variance of W_r obtained above are valid only so long as $(n-2r+2) \geqslant 0$ or $r \leqslant (n/2+1)$ because when r exceeds $(n/2+1)$ the equations given in (3.1) do not hold good. Consequently separate formulæ have to be developed to cover this case.

When $r > (n/2+1)$, for convenience we shall take the size of the blocks to be $r = (n-R)$. Let $X_1, X_2, \cdots X_{n-n-1}$; be defined by a set of equations similar to (3.1) the coefficients of which are represented by the following pattern:—

$$\left.\begin{array}{l}
1\ 2\ 3\cdots R,\ (R+1),\ (R+1)\cdots\cdot \\
\qquad (R+1),\ R,\ (R-1)\cdot\cdot 4\ 3\ 2\ 1 \\
1\ 2\ 3\cdots R\,(R+1),\ (R+1)\cdots\cdot \\
\qquad R\,(R-1),\ (R-2)\cdot\cdot 3\ 2\ 1 \\
1\ 2\ 3\cdots R\,(R+1),\ (R+1)\cdots\cdot \\
\qquad (R-1),\ R,\ (R-3)\cdot\cdot 2\ 1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
1\ 2\ 3\cdots R\,(R+1),\ (R+1)\ldots\ldots\ldots\ldots 3\ 2\ 1 \\
1\ 2\ 3\cdots R\ (R+1)\ R\ldots\ldots\ldots\ldots 2\ 1 \\
1\ 2\ 3\cdots R\ R\ (R-1)\ldots\ldots\ldots\ldots 1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
1\ 2\ 3\cdots 3\ 3\ 2\ 1 \\
1\ 2\ 2\cdots 2\ 2\ 1 \\
1\ 1\ \cdots\ 1\ 1
\end{array}\right\} \tag{3.16}$$

There are $(n - R - 1)$ rows in all. The first row contains $(n - 1)$ values having $(n - 2R - 1)$ central values equal to $(R + 1)$; the second $(n - 2)$ having $(n - 2R - 2)$ central values equal to $(R + 1)$, and so on; the last row having $(R + 1)$ values all of them being equal to 1.

Proceeding on the same lines as in the previous case, we obtain

$$E\,(W_{n-R}) = (R+1)\binom{n-R}{2}\,a_2,$$

and

$$\left.\begin{aligned}
\kappa_2\,(W_{n-R}) = \frac{1}{60}\,&(R+1)\,[5\,\{6n^2\,(R+1) \\
&- 2n\,(8R^2+13R+3) \\
&+ R\,(11R^2+25R+12)\}\,(a_2-a_2^2) \\
&+ 2\,\{10n^3\,(R+1) - 30n^2\,(R+1)^2 \\
&+ 5n\,(6R^3+20R^2+20R+4) \\
&- R\,(11R^3+54R^2+81R+34)\} \\
&\times (a_3-a_2^2) + \{20n^3\,(R+1) \\
&- 10n^2\,(7R^2+14R+6)+10n\,(8R^3 \\
&+ 25R^2+22R+4) - R\,(29R^3 \\
&+ 131R^2+184R+76\}\,(a_2+a_3 \\
&- 2a_2^2)]
\end{aligned}\right\} \tag{3.17}$$

(*b*) *Statistics* T_r.—Let T_r stand for the total number of positive or negative differences between the pairs of observations i and j such that $j - i \leqslant (r - 1)$ and each difference occurs once only. Then taking the case of $r \leqslant (n/2 + 1)$ and as in (*a*) above let $X_1, X_2 \cdots X_{r-1}$ be defined as follows:—

$$\left.\begin{aligned} X_1 &= (12) + (23) + (34) + \cdots\cdots + (n-1, n) \\ X_2 &= (13) + (24) + (35) + \cdots\cdots + (n-2, n) \\ X_3 &= (14) + (25) + (36) + \cdots\cdots + (n-3, n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ X_{r-1} &= (1r) + (2, r+1) + (3, r+2) + \cdots\cdots \\ &\quad + (n - r + 1, n), \end{aligned}\right\} \quad (3.18)$$

where, as before, (ij) takes values 1 or 0 according as $(x_i - x_j)$ is positive or otherwise.

Proceeding on the same lines as in the case of W_r we get

$$\left.\begin{aligned} E(T_r) &= \tfrac{1}{2}(r-1)(2n-r)\,a_2 \\ \kappa_2(T_r) &= \tfrac{1}{6}(r-1)[3(2n-r)(a_2 - a_2^2) \\ &\quad + 12(r-1)(n-r)(a_3 - a_2^2) \\ &\quad + 2(r-2)(3n-2r)(a_2 + a_3 - 2a_2^2)] \end{aligned}\right\} \quad (3.19)$$

When $r > (n/2 + 1)$, say equal to $(n - R)$, we have the following results for the mean and the variance of T_{n-R}:

$$\left.\begin{aligned} E(T_{n-R}) &= \tfrac{1}{2}(n - R - 1)(n + R)\,a_2 \\ \kappa_2(T_{n-R}) &= \tfrac{1}{6}[3(n-R-1)(n+R)(a_2 - a_2^2) \\ &\quad + 2\{n(n-1)(n-2) - 2R(R-1) \\ &\quad \times (R+1)\}(a_3 - a_2^2) + 2(n - R \\ &\quad - 1)(n - R - 2)(n + 2R) \\ &\quad \times (a_2 + a_3 - 2a_2^2)] \end{aligned}\right\} \quad (3.20)$$

B. Positive and negative differences

(*a*) *Statistics* W_r'.—It may be noted that the total number of positive and negative differences between pairs of observations in blocks of length r is also equal to the number of times that pairs of observations of different kinds, like i and j, occur in the distribution. In this case (ij) defined earlier will assume the value 1 if $|x_i - x_j|$ is not equal to zero and 0 otherwise. Let W_r' represent the total number of positive and negative differences obtained from the $(n - r + 1)$ blocks in the given sequence. When $r \leqslant (n/2 + 1)$, the first two cumulants of this distribution can be evaluated by using the results obtained in sub-section (*a*) of (A) above as follows:—

Replace a_2 by $2a_2$; $(a_2 - a_2^2)$ by $(2a_2 - 4a_2^2)$ and each of $(a_3 - a_2^2)$ and $(a_2 + a_3 - 2a_2^2)$ by $(a_2 + 3a_3 - 4a_2^2)$ respectively in the expressions for $E(W_r)$ and $\kappa_2(W_r)$ because in this case also the same types of configurations ⌒; ⌓, ⌒⌒, ⌒ and ⌒ would be involved with the above expectations. On making these substitutions

$$\left.\begin{aligned} E(W_r') &= (n - r + 1)\, r\, (r - 1)\, a_2 \\ \kappa_2(W_r') &= \tfrac{1}{6} r (r - 1) [\{2n(2r - 1) - r(5r \\ &\quad - 7)\}(a_2 - 2a_2^2) + \{(r - 2)(4r^2 \\ &\quad - 5r + 3) + 2(n - 2r + 2)(3r^2 \\ &\quad - 5r + 1)\}(a_2 + 3a_3 - 4a_2^2)] \end{aligned}\right\} \quad (3.21)$$

When $r > (n/2 + 1)$ and equal to $(n - R)$ say, we get the following values for the mean and the variance:—

$$\left.\begin{aligned} E(W'_{n-R}) &= (R + 1)(n - R)(n - R - 1)\, a_2 \\ \kappa_2(W'_{n-R}) &= \tfrac{1}{6}(R + 1)[\{6n^2(R+1) - 2n(8R^2 \\ &\quad + 13R + 3) + R(11R^2 + 25R \\ &\quad + 12)\}(a_2 - 2a_2^2) + \{6n^3(R + 1) \\ &\quad - 2n^2(10R^2 + 20R + 9) + 2n(11R^3 \\ &\quad + 35R^2 + 32R + 6) - R(8R^3 \\ &\quad + 37R^2 + 53R + 22)\}(a_2 + 3a_3 \\ &\quad - 4a_2^2)] \end{aligned}\right\} \quad (3.22)$$

(*b*) *Statistics* T_r'.—In this case the formulæ reduce to the following when $r \leqslant (n/2 + 1)$:—

$$\left.\begin{aligned} E(T_r') &= (r - 1)(2n - r)\, a_2 \\ \kappa_2(T_r') &= \tfrac{1}{3}(r - 1)[\{3(2n - r)(a_2 - 2a_2^2)\} \\ &\quad + 2\{3n(2r - 3) - r(5r - 7)\}(a_2 \\ &\quad + 3a_3 - 4a_2^2)] \end{aligned}\right\} \quad (3.23)$$

When $r > (n/2 + 1)$ equal to $(n - R)$ say, we have

$$\left.\begin{aligned} E(T'_{n-R}) &= (n - R - 1)(n + R)\,a_2 \\ \kappa_2(T'_{n-R}) &= \tfrac{1}{3}\,[3\,(n-R-1)(n+R)(a_2-2a_2^2) \\ &\quad + \{3n^3-9n^2-6n\;(R^2+R-1)+2R \\ &\quad \times (R+1)(R+5)\}\;(a_2+3a_3-4a_2^2)] \end{aligned}\right\} \quad (3.24)$$

The corresponding values for non-free sampling can be evaluated by making the substitutions mentioned in (2.10).

In the above discussion we have not obtained the higher cumulants which will give an idea of the nature of distributions. It can be shown fiom considerations discussed in a previous paper (1952) that for all the *statistics* dealt in this paper the cumulants are linear functions in n when $r < n/2 + 1$ and the highest degree of r in the t-th cumulant associated with n will be $(2t + 1)$ for W and W' and $(t + 1)$ for T and T'. It follows from this that

$$\left.\begin{aligned} \gamma_{t-2}(W \text{ or } W') &= \frac{\kappa_t}{\kappa_2^{t/2}} \sim 0\left(\frac{1}{n}, \frac{1}{r}\right)^{t/2-1} \\ \gamma_{t-2}(T \text{ or } T') &= \frac{\kappa_t}{\kappa_2^{t/2}} \sim 0\left(\frac{1}{n}, \frac{1}{r}\right)^{t/2-1} \end{aligned}\right\} \quad (3.25)$$

and they tend to zero as n tends to infinity for any value of $r < n/2 + 1$. A similar argument holds good for $r > n/2 + 1$. Hence the distributions of all the statistics considered in this paper tend to the normal form.

It may further be observed that these statistics are consistent both in the usual sense and also in the sense defined by Wald and Wolfowitz namely that the probability of rejecting the null hypothesis when it is false should approach unity as the sample size tends to infinity. As regards the former, it can be established with the help of the Techebycheff's Inequality and the latter by using the technique of Mann and Whitney in a similar manner as has been done in an earlier paper (1954).

(c) *Number of zero differences and covariances between the number of positive and negative differences.*—It may be noted that the total number of positive and negative differences together with the number of zero differences is constant for a given sequence of observations and therefore we do not gain anything by discussing the distribution of zeroes.

It may, however, be added that the covariance for positive and negative differences would be helpful in devising a comprehensive method

of testing the randomness of a sequence of observations. Therefore, the covariances for positive and negative differences are given below:—
When $r \leqslant (n/2 + 1)$, we have

$$\left.\begin{aligned}
&\operatorname{cov}\{W_r(+), W_r(-)\} \\
&\quad = \frac{r(r-1)}{60}[\{15nr(r-1) - (21r^3 - 34r^2 \\
&\qquad + 11r + 6)\}a_2 + \{5n(9r^2 - 17r + 4) \\
&\qquad - (59r^3 - 156r^2 + 99r + 14)\}a_3 \\
&\qquad - 5\{2n(6r^2 - 8r + 1) - (16r^3 - 33r^2 \\
&\qquad + 15r + 4)\}a_2^2] \\
&\operatorname{cov}\{T_r(+), T_r(-)\} \\
&\quad = \frac{(r-1)}{6}[\{6n(r-1) - 6r(r-1)\}a_2 \\
&\qquad + \{6n(3r-5) - 2r(7r-11)\}a_3 \\
&\qquad - (4r-5)(6n-5r)a_2^2]
\end{aligned}\right\} \quad (3.26)$$

When $r > (n/2 + 1)$ and equal to $(n - R)$, we get

$$\left.\begin{aligned}
&\operatorname{cov}\{W_{n-R}(+), W_{n-R}(-)\} \\
&\quad = \frac{(R+1)}{60}[\{10n^3(R+1) - 30n^2(R+1)^2 \\
&\qquad + 10n(3R^3 + 10R^2 + 10R + 2) \\
&\qquad - R(11R^3 + 54R^2 + 81R + 34)\}a_2 \\
&\qquad + \{50n^3(R+1) - 10n^2(17R^2 + 34R \\
&\qquad + 15) + 10n(19R^3 + 60R^2 + 54R + 10) \\
&\qquad - R(69R^3 + 316R^2 + 449R + 186)\}a_3 \\
&\qquad - \{60n^3(R+1) - 10n^2(20R^2 + 37R \\
&\qquad + 15) + 10n(22R^3 + 62R^2 + 51R + 9) \\
&\qquad - R(80R^3 + 315R^2 + 405R + 160)\}a_2^2] \\
&\operatorname{cov}\{T_{n-R}(+), T_{n-R}(-)\} \\
&\quad = \tfrac{1}{6}[\{n(n-1)(n-2) - 2R(R-1)(R+1)\}a_2 \\
&\qquad + \{5n^3 - 15n^2 - 2n(6R^2 + 6R - 5) \\
&\qquad + 6R(R^2 + 4R + 3)\}a_3 - \{6n^3 - 15n^2 \\
&\qquad - 3n(4R^2 + 4R - 3) + R(4R^2 + 21R \\
&\qquad + 17)\}a_2^2]
\end{aligned}\right\} \quad (3.27)$$

4. Applications

The statistics W_r, W_r', T_r, T_r' considered in the previous sections can be used for testing (1) whether a given sequence of observations is random or not and (2) whether two or more samples can be treated as samples from the same population. The test for randomness of a given sequence consists in noting the observed values of W's or T's and comparing them with their expected values on the basis of their variances on the assumption that the standardised deviates of the statistics are distributed normally. As regards (2) the procedure is to pool together the various samples and arrange them in ascending or descending order indicating the samples to which these observations belong by designating the samples by 1, 2, etc...... In this set-up we obtain a sequence of observations for the characters 1, 2, etc. We then examine whether this sequence is random or not, by the statistic W, W', T or T' for the characters 1, 2, etc. It may be noted that in arranging the samples in this manner it will not be possible to have a unique arrangement when the samples belong to discontinuous populations. In this case we shall take the average of the observed W, W', T or T', as the case may be, for the different possible arrangements. Alternatively, the test may be applied by considering the first part of the sequence as Sample I and the second part as Sample II, and r being equal to $n_1 + h$, where n_1 is the size of the first sample and $h \leqslant n_2$, the size of the second sample.

A more comprehensive test than the one given above can be had by examining the significance of the difference between the observed number x of positive and y of negative differences obtained for the statistic W_r (or T_r) on the basis of the following bivariate *statistic*

$$\frac{1}{1-\rho^2}\left\{\frac{(x-m)^2}{\sigma_x^2}+\frac{(y-m)^2}{\sigma_y^2}-2\rho\,\frac{(x-m)(y-m)}{\sigma_x\sigma_y}\right\} \qquad (4.1)$$

where x and y, as already explained, stand for the observed number of positive and negative differences in W_r (or T_r) in the given sequence; m and σ^2 for the mean and the variance for W_r (or T_r) and ρ is the correlation coefficient between $W_r(+)$ and $W_r(-)$ [or $T_r(+)$ and $T_r(-)$].

Now the question as to which statistics should be used in actual practice can be decided only after examining their powers for different alternatives and their asymptotic relative efficiencies. These aspects are considered in the next section.

5. Power and Efficiency of the Statistics

A number of non-parametric tests has been developed during the past two decades for testing the randomness of a given sequence of observations and the homogeneity of two or more samples. The efficiency of these tests can be studied in general by examining the power curves for different types of alternatives. The possible alternatives here, unlike the parametric tests, are many and it is possible that a test which is efficient for one type of alternative may not be so for another type. The alternatives mostly considered are either normal or normal regression. In normal alternatives the distribution of the parent population is assumed to be normal while in the other the assumption is that

$$y_i = \alpha + \beta X_i + \epsilon_i \qquad (i = 1, 2, \cdots n) \tag{5.1}$$

where ϵ_i is distributed normally with zero mean and unit variance.

As the calculation of the actual powers is very cumbersome, Walsh (1946) suggested that the relative efficiency of the tests can be obtained by comparing the sizes of the samples required for a given power against a given type of alternative. Two power curves are considered to be equivalent if their average height is the same. Dixon (1953) has pointed out that the equivalence by averaging process disguises the differences in the shape of the curves. He has, therefore, suggested that it would be more realistic to define a power efficiency function which would give the power efficiency for each alternative of a given type. Following Walsh, Pitman (1948) has defined the asymptotic relative efficiency of two tests by taking in the limit, under certain conditions, the reciprocal of the ratio of sample sizes required to attain the same power against the same alternative at $\theta = \theta_0 + \epsilon$ as ϵ tends to zero and n to infinity. Mood (1954) shows that the asymptotic relative efficiency as defined by Pitman is the same as the ratio of the changes in power as θ changes from θ_0 to $\theta_0 + \epsilon$ when $|\theta - \theta_0| \leqslant 1/\sqrt{n}$.

It may be noted that Pitman's result follows directly from that of Wald (1945) given in connection with his investigations on sequential analysis. The size of the sample required for a normal distribution for specified (α, β, θ_1, θ_1) is given by

$$n = \frac{(\lambda_1 - \lambda_0)^2}{(\theta_1 - \theta_0)^2} \tag{5.2}$$

where λ_0 and λ_1 are the standardized deviates for the hypothesis $\theta = \theta_0$ and θ_1 respectively for the probabilities $(1 - \alpha)$ and β. When θ_1 tends to θ_0 the above reduces to

$$n = \frac{1}{\sigma^2}\left(\frac{d\mu}{d\theta}\right)^2 \tag{5.3}$$

We shall now examine the power and the relative efficiency of the tests developed in this paper for different values of r and compare them with the Wilcoxon's or Mann and Whitney's test which corresponds to W_n or T_n of the binomial case for $r = n$. We shall also investigate the relative efficiency of the various statistics for testing the randomness of sequences belonging to continuous populations. It may be observed that, in general, the *relative* efficiency, as defined by Pitman, of the statistics considered in this paper for different values of r and of others of allied forms can equally be ascertained by taking the reciprocal of the squares of the coefficients of variation of the statistics. This can be seen from the fact that the expected values of the statistics are of the form

$$E\ (W_r \text{ say}) = k\,(r)\ a_2$$

where $k\,(r)$ is a function of r and $a_2 = \Sigma\, p_i p_j$. Then

$$\frac{dE}{d\theta} = k\,(r)\,\frac{da_2}{d\theta} \tag{5.4}$$

on the assumption that $da_2/d\theta$ is the same for all values of r, it can be easily seen that the relative efficiency, which depends on

$$\frac{1}{\sigma^2}\left(\frac{dE}{d\theta}\right)^2 = \frac{k^2\,(r)}{\sigma^2}\left(\frac{da_2}{d\theta}\right)^2, \tag{5.5}$$

is directly proportional to the square of the reciprocal of the coefficient of variation. In view of this fact, we shall be content by examining the relative efficiency of the statistics on the basis of the squares of their coefficients of variation.

The efficiency of the different statistics developed in this paper has been examined firstly by calculating their powers for different hypotheses and alternatives and secondly by finding the squares of their coefficients of variation. The powers calculated for $n = 100$ and 200 and for different values of p's are tabulated in Tables II to IV. The powers for different values of r for H_0: $(p = \cdot 5$ and $q = \cdot 5)$ and $(p = \cdot 2$ and $q = \cdot 8)$ are shown in Figs. 1 to 6 for some alternatives. In these graphs the curves I, II, III and IV refer to the statistics T_r, W_r, W_r' and T_r' respectively. A study of the graphs and tables giving the powers of the various statistics for $n = 100$ leads to the following conclusions:—

(i) When the null hypothesis is $p = \cdot 5$, $q = \cdot 5$, we find that the statistics W_r, which are based on all the possible positive (or negative) differences taken from $(n - r + 1)$ blocks each consisting of r contiguous observations, are in general more powerful than T_r in which the differences between any two observations occur once only. The

TABLE II

Powers of different tests for various alternatives in comparing two samples

$n = 100$; $H_0 - p = q = 0 \cdot 5$

Hypothesis	r=2	r=5	r=10				r=15			r=18		r=20			r=25
	W_r or T_r	T_r	W_r	W_r'	T_r	T_r'	W_r	W_r'	T_r	W_r	W_r'	W_r	T_r	T_r'	W_r'
$H_0 - p = \cdot 50$, $q = \cdot 50$	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500
$H_1 - p = \cdot 45$, $q = \cdot 55$	0·0533	0·0676	0·0802	0·0886	0·0786	0·0988	0·0866	0·1096	0·0785	0·0872	0·1217	0·0867	0·0749	0·1497	0·1483
$H_2 - p = \cdot 40$, $q = \cdot 60$	0·0767	0·1673	0·2329	0·2728	0·2269	0·3180	0·2608	0·3509	0·2254	0·2626	0·3868	0·2589	0·2056	0·4651	0·4496
$H_3 - p = \cdot 35$, $q = \cdot 65$	0·1623	0·4185	0·5318	0·5880	0·5258	0·6453	0·5671	0·6695	0·5217	0·5663	0·6992	0·5603	0·4874	0·7671	0·7426
$H_4 - p = \cdot 30$, $q = \cdot 70$	0·3655	0·7314	0·8135	0·8484	0·8125	0·8815	0·8319	0·8871	0·8082	0·8293	0·8990	0·8242	0·7824	0·9314	0·9141
$H_5 - p = \cdot 25$, $q = \cdot 75$	0·6706	0·9315	0·9580	0·9684	0·9589	0·9781	0·9621	0·9775	0·9569	0·9604	0·9799	0·9571	0·9477	0·9885	0·9824
$H_6 - p = \cdot 20$, $q = \cdot 80$	0·9169	0·9931	0·9961	0·9973	0·9964	0·9984	0·9964	0·9981	0·9961	0·9960	0·9983	0·9956	0·9948	0·9993	0·9984
$H_7 - p = \cdot 15$, $q = \cdot 85$	49925*	98752*	99347*	99610*	99458*	99822*	99338*	99726*	99362*	99208*	99741*	99071*	990[illegible]2*	99925*	99725*
$H_8 - p = \cdot 10$, $q = \cdot 90$	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

TABLE II—*Contd.*

Hypothesis	$r=30$		$r=35$	$r=40$		$r=45$	$r=50$	$r=80$		$r=90$	$r=100$	
	W_r'	T_r'	W_r'	W_r'	T_r'	W_r'	T_r'	W_r'	T_r'	W_r'	W_r or T_r	W_r' or T_r'
H_0 – $p=\cdot 50$, $q=\cdot 50$	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500
H_1 – $p=\cdot 45$, $q=\cdot 55$	0·1655	0·1910	0·1813	0·1993	0·2233	0·2083	0·2475	0·2641	0·2851	0·2784	0·0530	0·2920
H_2 – $p=\cdot 40$, $q=\cdot 60$	0·4821	0·5414	0·5078	0·5284	0·5872	0·5453	0·6170	0·6260	0·6591	0·6469	0·0719	0·6661
H_3 – $p=\cdot 35$, $q=\cdot 65$	0·7615	0·8127	0·7751	0·7853	0·8369	0·7935	0·8520	0·8486	0·8741	0·8639	0·1378	0·8775
H_4 – $p=\cdot 30$, $q=\cdot 70$	0·9197	0·9467	0·9233	0·9259	0·9543	0·9280	0·9592	0·9544	0·9670	0·9619	0·2960	0·9681
H_5 – $p=\cdot 25$, $q=\cdot 75$	0·9849	0·9912	0·9833	..	0·9926	0·9835	0·9935	0·9916	0·9951	..	0·5589	0·9953
H_6 – $p=\cdot 20$, $q=\cdot 80$	0·9984	0·9994	0·9983	..	0·9995	0·9982	0·9996	0·9993	0·9997	..	0·8332	0·9997
H_7 – $p=\cdot 15$, $q=\cdot 85$	99689*	99945*	99642*	..	99955*	99566*	99962*	99911*	99978*	..	0·9769	99980*
H_8 – $p=\cdot 10$, $q=\cdot 90$	1·0000	1·0000	1·0000	..	1·0000	1·0000	1·0000	1·0000	1·0000	..	0·9997	1·0000

* Prefix 0·99.

TABLE III

Powers of different tests for various alternatives in comparing two samples

$n = 100$; H_0—$p = \cdot 20$; $q = \cdot 80$.

Hypothesis	$r=2$	$r=5$		$r=6$	$r=10$			$r=15$			$r=20$		
	W_r or T_r	W_r'	T_r	W_r	W_r	W_r'	T_r	W_r	W_r'	T_r	W_r	T_r	T_r'
H_1 — $p = \cdot 50$, $q = \cdot 50$	0·9046	76483*	81809*	85311*	98515*	99819*	98626*	99348*	1·0000	98208*	98803*	94723*	1·0000
H_2 — $p = \cdot 45$, $q = \cdot 55$	0·8847	0·9940	0·9950	0·9957	0·9990	0·9997	0·9991	0·9993	99571*	89117*	89231*	0·9977	99997*
H_3 — $p = \cdot 40$, $q = \cdot 60$	0·8114	0·9610	0·9644	0·9653	0·9784	0·9858	0·9809	0·9788	0·9899	0·9785	0·9729	0·9712	0·9970
H_4 — $p = \cdot 35$, $q = \cdot 65$	0·6530	0·8088	0·8149	0·8131	0·8296	0·8471	0·8408	0·8218	0·8491	0·8329	0·8023	0·8163	0·8914
H_5 — $p = \cdot 30$, $q = \cdot 70$	0·4048	0·4872	0·4916	0·4872	0·4901	0·5016	0·5030	0·4775	0·4951	0·4949	0·4594	0·4823	0·5344
H_6 — $p = \cdot 25$, $q = \cdot 75$	0·1572	0·1701	0·1712	0·1692	0·1675	0·1699	0·1718	0·1629	0·1664	0·1693	0·1605	0·1662	0·1769
H_0 — $p = \cdot 20$, $q = \cdot 80$	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500
H_7 — $p = \cdot 15$, $q = \cdot 85$	0·1957	0·2450	0·2471	0·2455	0·2485	0·2539	0·2536	0·2441	0·2523	0·2503	0·2370	0·2447	0·2685
H_8 — $p = \cdot 10$, $q = \cdot 90$	0·6957	0·7818	0·7854	0·7808	0·7809	0·7896	0·7922	0·7691	0·7826	0·7853	0·7528	0·7750	0·8122

TABLE III—*Contd.*

Hypothesis	$r=25$	$r=30$		$r=45$			$r=50$	$r=80$		$r=100$	
	W_r'	W_r'	T_r	W_r	W_r'	T_r	T_r'	W_r'	T_r'	W_r or T_r	W_r' or T_r'
$H_1 - \begin{matrix} p=\cdot 50 \\ q=\cdot 50 \end{matrix}$	1·0000	1·0000	57916*	0·9662	1·0000	0·9694	1·0000	1·0000	1·0000	0·7955	1·0000
$H_2 - \begin{matrix} p=\cdot 45 \\ q=\cdot 55 \end{matrix}$	99921*	99938*	0·9901	0·9446	99932*	0·9536	1·0000	1·0000	1·0000	0·7684	1·0000
$H_3 - \begin{matrix} p=\cdot 40 \\ q=\cdot 60 \end{matrix}$	0·9898	0·9884	0·9455	0·8492	0·9824	0·8813	0·9985	0·9959	0·9991	0·6804	0·9992
$H_4 - \begin{matrix} p=\cdot 35 \\ q=\cdot 65 \end{matrix}$	0·8323	0·8195	0·7734	0·6360	0·7829	0·6972	0·8998	0·8601	0·9144	0·5222	0·9163
$H_5 - \begin{matrix} p=\cdot 30 \\ q=\cdot 70 \end{matrix}$	0·4699	0·4561	0·4587	0·3554	0·4226	0·4089	0·5374	0·4899	0·5563	0·3144	0·5588
$H_6 - \begin{matrix} p=\cdot 25 \\ q=\cdot 75 \end{matrix}$	0·1579	0·1481	0·1599	0·1318	0·1441	0·1497	0·1766	0·1624	0·1822	0·1284	0·1829
$H_0 - \begin{matrix} p=\cdot 20 \\ q=\cdot 80 \end{matrix}$	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500
$H_7 - \begin{matrix} p=\cdot 15 \\ q=\cdot 85 \end{matrix}$	0·2433	0·2383	0·2316	0·1913	0·2263	0·2087	0·2701	0·2517	0·2778	0·1550	0·2788
$H_8 = \begin{matrix} p=\cdot 10 \\ q=\cdot 90 \end{matrix}$	0·7603	0·7481	0·7499	0·6398	0·7174	0·7025	0·8130	0·7762	0·8263	0·5569	0·8281

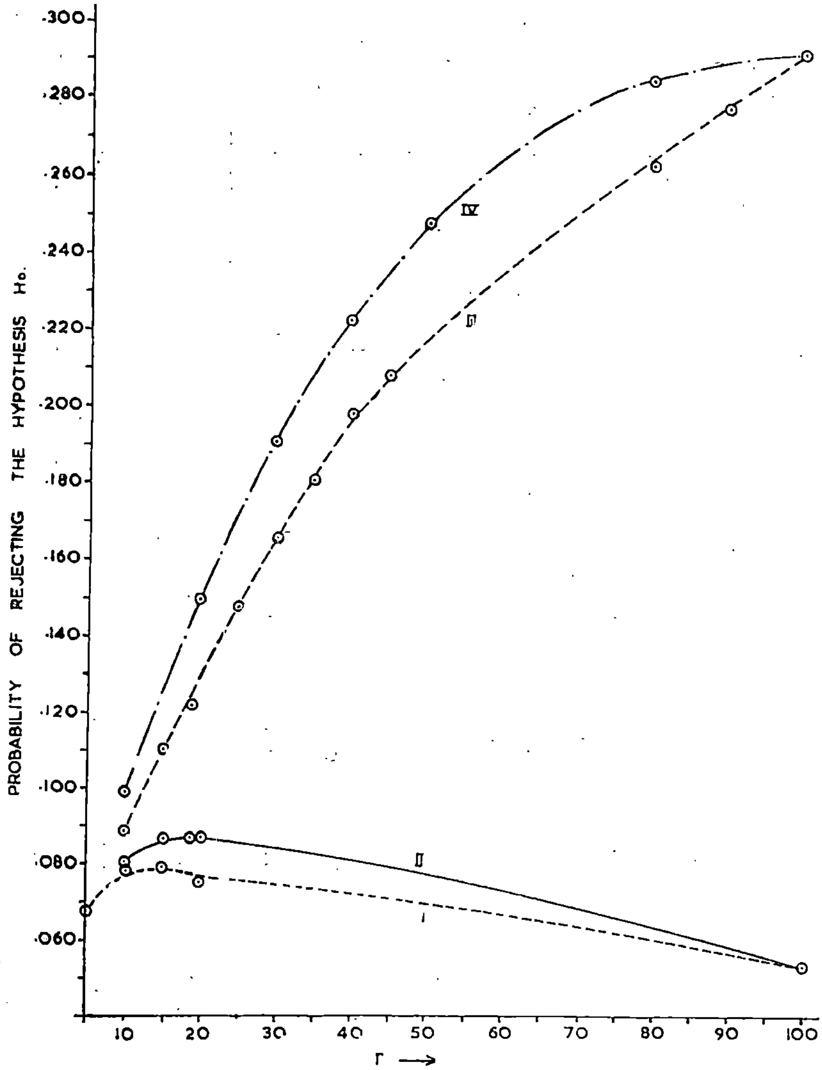
* Prefix 0·99.

TABLE IV

Powers of different tests for various alternatives in comparing two samples

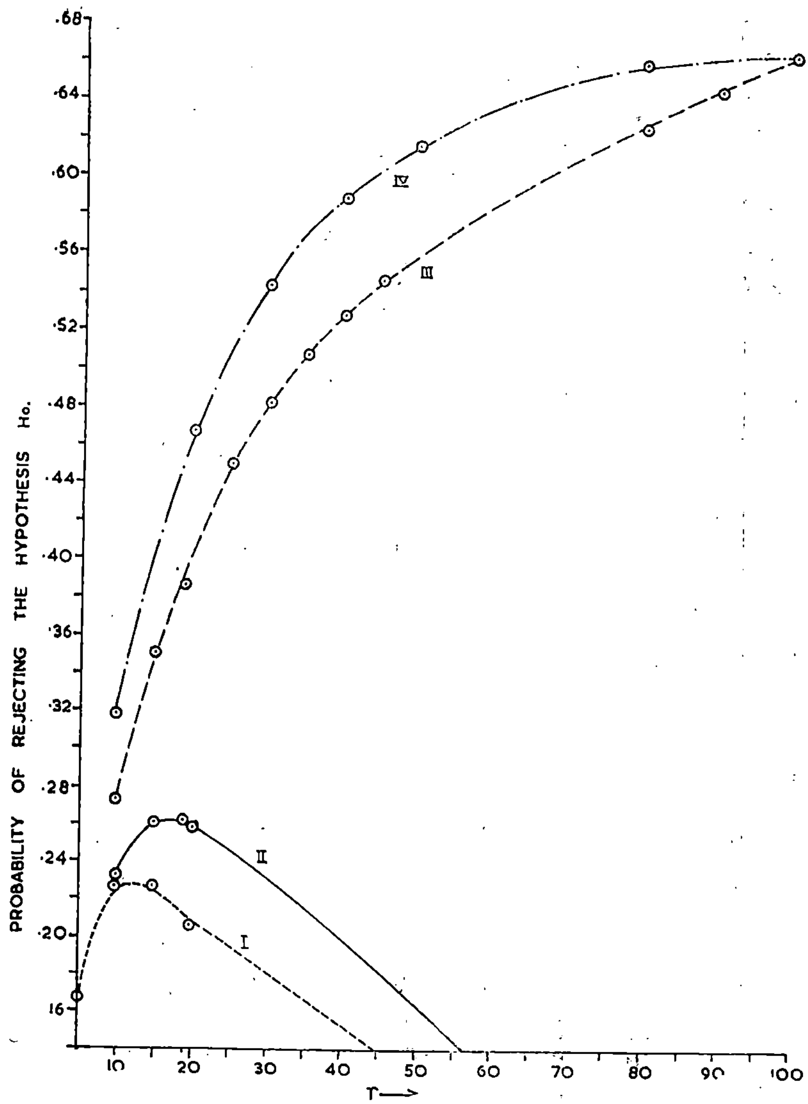
$n = 200;\ H_0 - p = q = \cdot 50$

Hypothesis	$r=2$	$r=10$	$r=20$	$r=30$	$r=40$		$r=60$	$r=80$	$r=200$	
	W_r or T_r	W_r	W_r	W_r	W_r	W_r'	W_r'	W_r'	W_r	W_r'
$H_0 - p=\cdot 50,\ q=\cdot 50$	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500	0·0500
$H_1 - p=\cdot 45,\ q=\cdot 55$	0·0545	0·0918	0·1156	0·1177	0·1101	0·2294	0·2877	0·3343	0·0539	0·4665
$H_2 - p=\cdot 40,\ q=\cdot 60$	0·0960	0·3488	0·4450	0·4476	0·4154	0·6686	0·7189	0·7485	0·0864	0·8493
$H_3 - p=\cdot 35,\ q=\cdot 65$	0·2595	0·7585	0·8277	0·8250	0·7988	0·9219	0·9327	0·9369	0·2116	0·9740
$H_4 - p=\cdot 30,\ q=\cdot 70$	0·6014	0·9638	0·9771	0·9754	0·9685	0·9909	0·9914	0·9913	0·4940	0·9982
$H_5 - p=\cdot 25,\ q=\cdot 75$	0·9124	83861*	90374*	88499*	82930*	96249*	95841*	95174*	0·8271	99476*
$H_6 - p=\cdot 20,\ q=\cdot 80$	64288*	99896*	99937*	99911*	99834*	99973*	99961*	99941*	0·9821	1·0000
$H_7 - p=\cdot 15,\ q=\cdot 85$	99946*	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	98251*	1·0000
$H_8 - p=\cdot 10,\ q=\cdot 90$	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

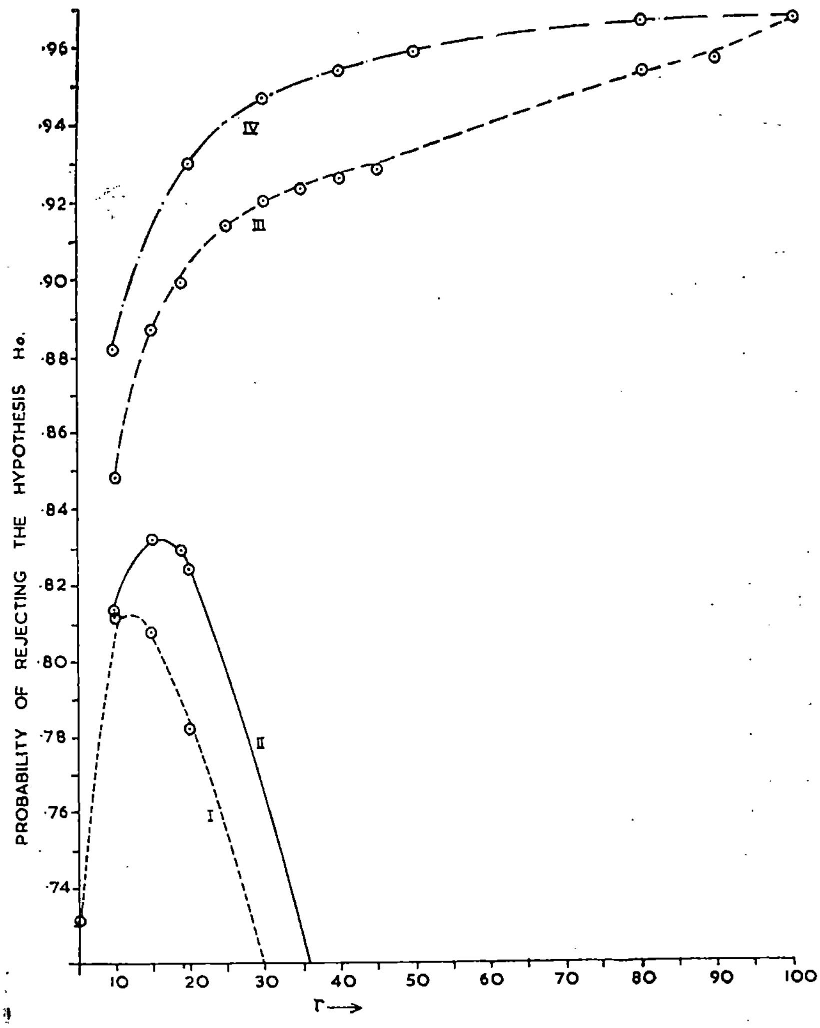


GRAPH 1. Power of the tests for different values of r, $n = 100$; H_0: $p_0 = \cdot 5$; H_1: $p_1 = \cdot 45$.

powers of both W_r and T_r increase with r, attain their maximum values and then gradually decrease. In fact, the power for even $r = 2$ is slightly more than that for $r = n$ and the two statistics W_r and T_r are identical in these two cases. Also, the value of the maximum power for W_r is more than that for T_r. The table below explains the position more clearly in regard to the maximum power.

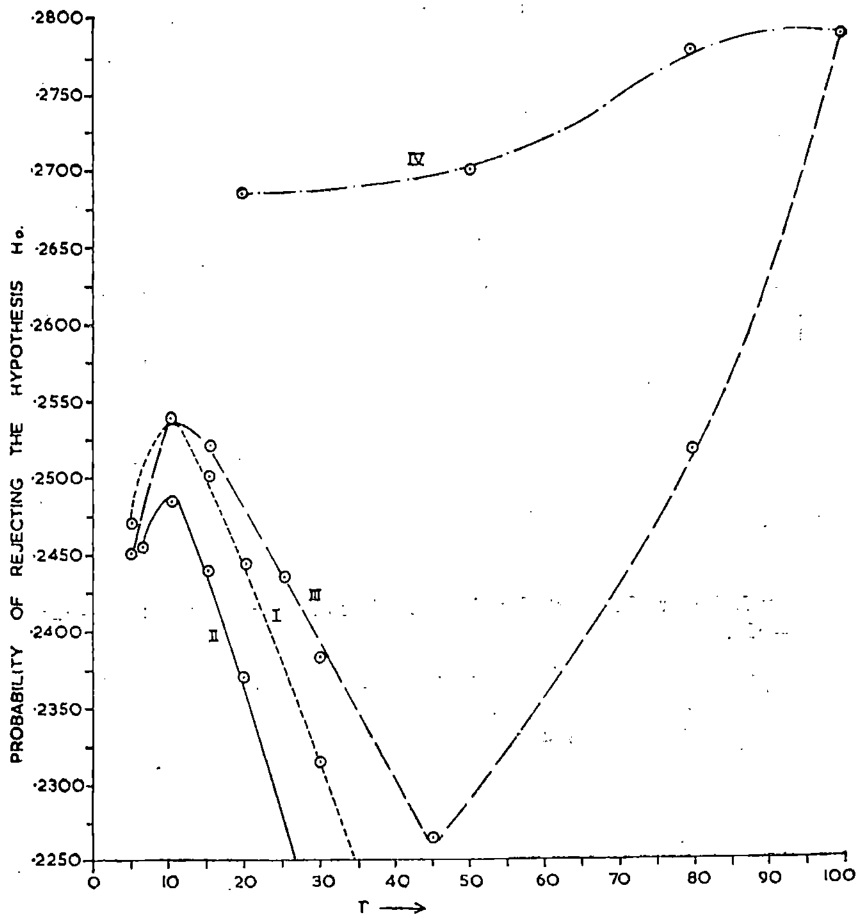


GRAPH 2. Power of the tests for different values of r, $n = 100$; H_0: $p_0 = \cdot 5$; H_1: $p_1 = \cdot 40$.

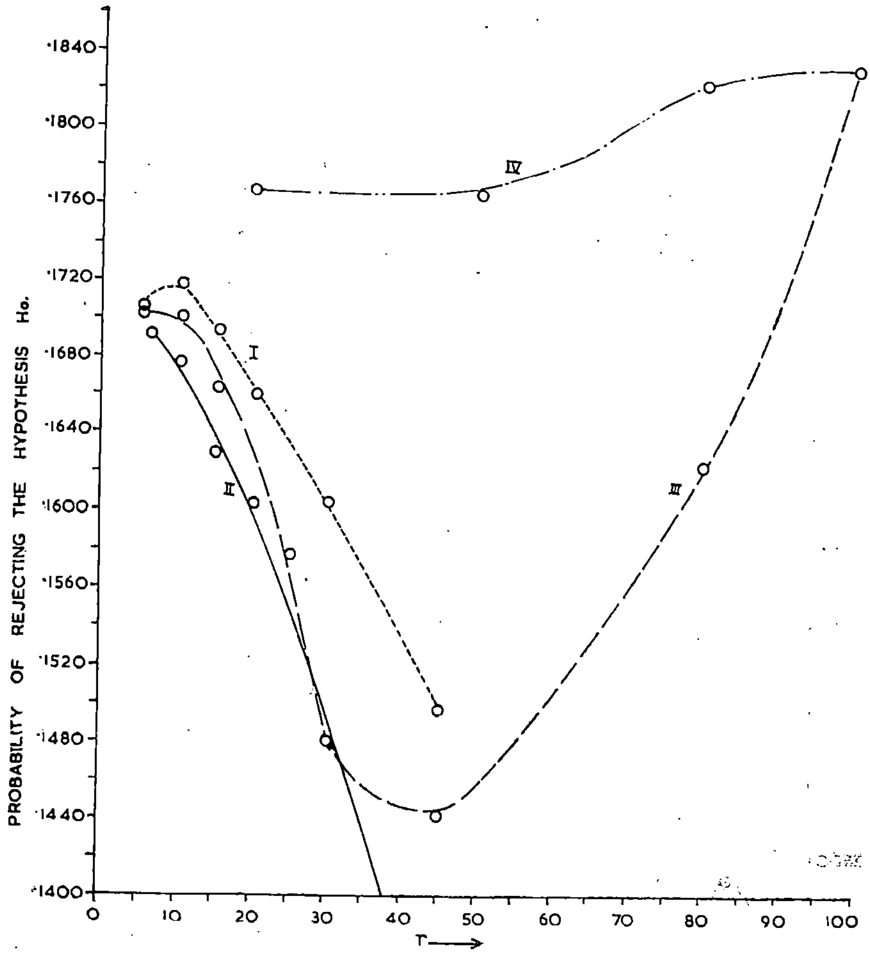


GRAPH 3. Power of the tests for different values of r, $n = 100$; H_0: $p_0 = \cdot 5$; H_1: $p_1 = \cdot 30$.

Alternative hypothesis H_1		Maximum power with the corresponding r			
		W_r		T_r	
p	q	Power	r	Power	r
·45	·55	·0872	18	·0786	10
·40	·60	·2626	18	·2269	10
·35	·65	·5671	15	·5258	10
·30	·70	·8319	15	·8125	10
·25	·75	·9621	15	·9589	10
·20	·80	·9964	15	·9964	10
·15	·85	·999935	10	·999946	10

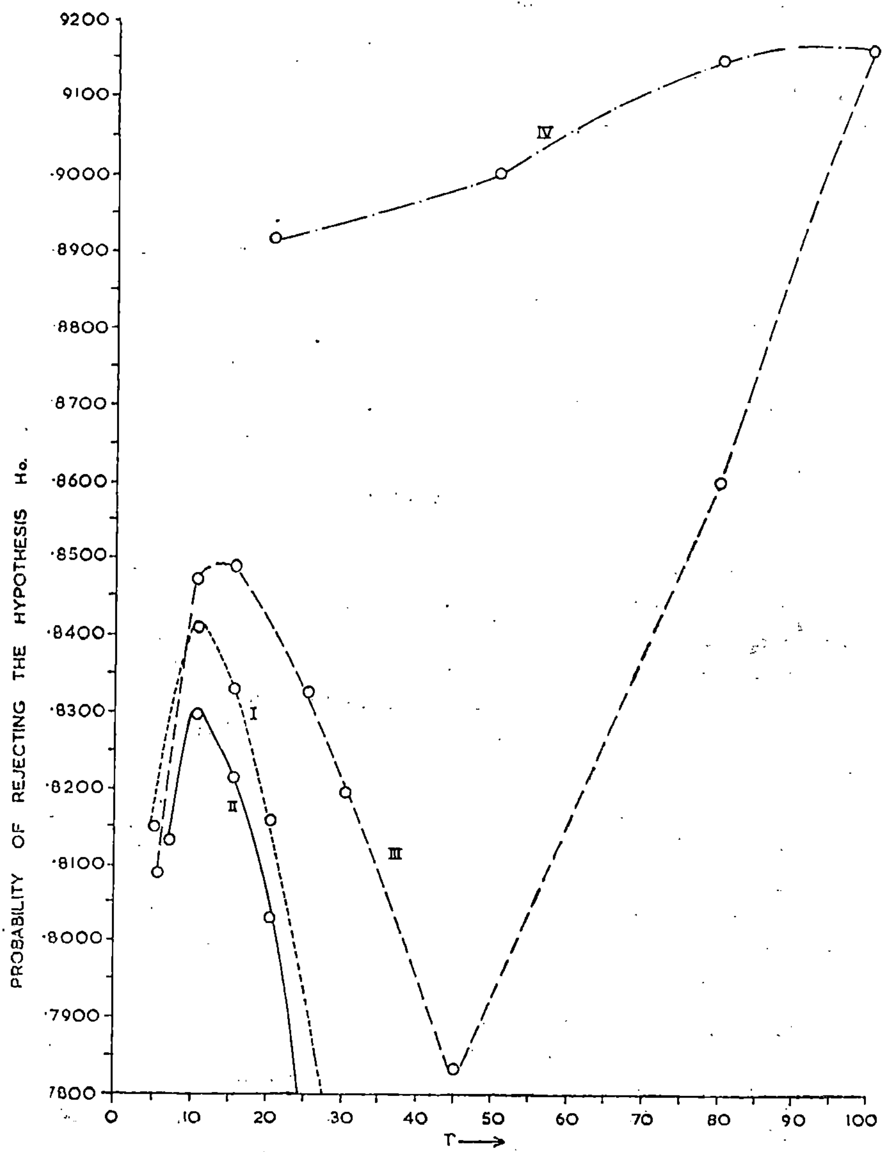


GRAPH 4. Power of the tests for different values of r, $n = 100$; H_0: $p_0 = \cdot 2$; H_1: $p_1 = \cdot 15$.



Graph 5. Power of the tests for different values of r, $n = 100$; H_0: $p_0 = \cdot 2$; H_1: $p_1 = \cdot 25$.

(ii) When the null hypothesis is $p = \cdot 2$, $q = \cdot 8$, we find that the statistics W_r are generally more powerful than T_r only when the alternative hypotheses ($p = \cdot 5$, $\cdot 45$ and $\cdot 40$) are far removed from H_0, r taking values 15 to 20, otherwise T_r exhibit greater power than W_r. The following table summarises the information given in Table III as regards maximum power.



GRAPH 6. Power of the tests for different values of r, $n = 100$; $H_0: p_0 = \cdot 2$; $H_1: p_1 = \cdot 35$.

Alternate hypothesis H_1		Maximum power with the corresponding r			
		W_r		T_r	
p	q	Power	r	Power	r
·50	·50	·999935	15	·99986	10
·45	·55	·9993	15	·9991	10
·40	·60	·9788	15	·9809	10
·35	·65	·8296	10	·8408	10
·30	·70	·4901	10	·5030	10
·25	·75	·1692	6	·1718	10
·15	·85	·2485	10	·2536	10
·10	·90	·7809	10	·7922	10

(iii) As regards T_r' its power uniformly increases as r increases in all the cases and reaches a maximum only at $r = n$. Regarding W_r' it would be noticed that its power, even though it increases with increasing r excepting for a few cases, is always less than that of T_r' with the only exception of $r = n$, when W_n' is equal to T_n'. However, the powers of both W_r' and T_r' are generally more than those of W_r and T_r. The results for $n = 200$ are almost similar to that of $n = 100$. In this case the power of W_r is maximum for r ranging from 20 to 30.

It would be seen that, in general, T_r' is more powerful than any of the other tests W_r', T_r and W_r. But, for testing the randomness of a sequence of observations from a continuous population, T_r' cannot be used as it becomes a constant quantity and hence its distribution does not exist. The appropriate test to be used in this situation is only W_r or T_r.

However, when we are concerned with testing the homogeneity of two samples from continuous distributions $f(x)$ and $g(x)$, it is possible to use T_r' and in that case T_n' would be the most powerful test for this purpose. It may be remarked that in this case T_n would correspond to Mann and Whitney's U-test which is closely related to Wilcoxon's T-test and for which the power, as compared to the t-test, has been shown to be equal to $3/\pi$ for normal distributions. Thus, the tests based on T_r' and also T_r and W_r appear to be more powerful than even the Wilcoxon's test and possibly t-test also.

As has already been explained, the asymptotic relative efficiency of two tests can be obtained by comparing the reciprocals of the squares of the coefficients of variation on the assumption that $da_2/d\theta$ is constant throughout the sequence. In view of this fact, it follows that the asymptotic relative efficiency of a test can be taken to be inversely proportional to its square of the coefficient of variation. Therefore, the test with the least coefficient of variation can be considered to be the best test. Tables V to VII give the squares of the coefficients of variation (*c.v.*) for the different tests considered in this paper for sequences from continuous as well as discontinuous distributions. From Table V it would be seen that for $n = 50$, 100 and infinity, the $(c.v.)^2$ for both W_r and T_r decreases with r, reaches a minimum and then steadily increases. However the fall and the increase in $(c.v.)^2$ is more rapid for T_r than for W_r in the beginning and the end, thus giving a smaller value of $(c.v.)^2$ for T_r than W_r. But in between the values of $(c.v.)^2$, are less for W_r than T_r, the former having the lowest value and therefore the statistic W_r is to be preferred to T_r in general. For $n = 50$ and 100, the values of r for which $(c.v.)^2$ is minimum are near about 10 and 15 respectively. It would further be seen that the ratios of the minimum to the maximum $(c.v.)^2$ for $n = 50$ and 100 vary from 5 to 8. Since the maximum $(c.v.)^2$ corresponds to the Mann and Whitney's or Wilcoxon's test, for which the relative efficiency is $3/\pi$ as compared to the t-test, it would follow that the efficiency of the tests developed in this paper appear to be more than the t-test. It may, however, be emphasized that the tests discussed here are not really comparable with the t-test because the latter completely ignores the order of occurrence of the observations while the W's and T's are based on the order or time of occurrence of the observations.

The tables showing the $(c.v.)^2$ for different values of p and q corresponding to discontinuous distributions confirm in general the findings of the power curves. The $(c.v.)^2$ for the statistics T_r and W_r' is minimum for $r = n$, both for $n = 100$ and 200. Examining W_r and T_r it would be seen that there is not much difference between the minimum values, though W_r has a lower minimum than T_r when p is near about $\cdot 5$. When p is far removed from $\cdot 5$, T_r is minimum. These minimum values occur at $r = 10$ or 15 for $n = 100$. When $n = 200$, this minimum occurs for $r = 20$ to 30.

Thus it would be seen that both from the points of view of power and asymptotic relative efficiency, as seen from $(c.v.)^2$, the tests based on W_r are superior to T_r when p is not far removed from $\cdot 5$. When p is near about $\cdot 2$, T_r is better. In cases where W_r' and T_r' can be

TABLE V

Squares of the coefficients of variation of W_r *and* T_r *for the continuous case*

Size of the Sequence (n)

50			100			$n \sim$ Infinity		
r	$(C.V.)^2$		r	$(C.V)^2$		r	$(C.V.)^2$	
	W_r	T_r		W_r	T_r		W_r	T_r
2	70·8	70·8	2	34·4	34·4	2	$3333 \cdot 3\frac{1}{n}$	$3333 \cdot 3\frac{1}{n}$
5	24·5	23·1	5	11·1	9·9	5	$1000 \cdot 0\frac{1}{n}$	$833 \cdot 3\frac{1}{n}$
10	17·3	19·8	10	6·4	6·5	10	$469 \cdot 1\frac{1}{n}$	$370 \cdot 4\frac{1}{n}$
25	41·0	44·0	15	5·4	6·6	100	$44 \cdot 7\frac{1}{n}$	$33 \cdot 7\frac{1}{n}$
40	90·0	83·1	20	5·6	7·6	$\frac{n}{20000}$	$0 \cdot 044\frac{1}{n}$	$0 \cdot 11\frac{1}{n}$
50	95·2	95·2	50	19·1	20·9	$\frac{n}{10000}$	$0 \cdot 089\frac{1}{n}$	$0 \cdot 22\frac{1}{n}$
			80	43·1	33·4	$\frac{n}{1000}$	$0 \cdot 89\frac{1}{n}$	$2 \cdot 2\frac{1}{n}$
			100	46·0	46·0	$\frac{n}{100}$	$9 \cdot 1\frac{1}{n}$	$22 \cdot 4\frac{1}{n}$
						$\frac{n}{50}$	$18 \cdot 5\frac{1}{n}$	$45 \cdot 5\frac{1}{n}$
						$\frac{n}{20}$	$49 \cdot 2\frac{1}{n}$	$116 \cdot 9\frac{1}{n}$
						$\frac{n}{15}$	$69 \cdot 4\frac{1}{n}$	$158 \cdot 5\frac{1}{n}$
						$\frac{n}{10}$	$109 \cdot 7\frac{1}{n}$	$246 \cdot 2\frac{1}{n}$
						$\frac{n}{5}$	$277 \cdot 8\frac{1}{n}$	$548 \cdot 7\frac{1}{n}$
						$\frac{n}{2}$	$1777 \cdot 8\frac{1}{n}$	$1975 \cdot 3\frac{1}{n}$
						$\frac{4n}{5}$	$4184 \cdot 0\frac{1}{n}$	$3896 \cdot 6\frac{1}{n}$
						n	$4444 \cdot 4\frac{1}{n}$	$4444 \cdot 4\frac{1}{n}$

TABLE VI

$(C.V.)^2$ *of different statistics for* n = 100

Hypothesis	$r=2$		$r=5$	$r=10$				$r=15$			$r=18$		$r=20$		
	W_r or T_r	W_r' or T_r'	T_r	W_r	W_r'	T_r	T_r'	W_r	W_r'	T_r	W_r	W_r'	W_r	T_r	T_r'
$p=\cdot 50$ $q=\cdot 50$	103	101	30	19	15	19	12	16	10	20	17	9	17	23	6
$p=\cdot 45$ $q=\cdot 55$	107	105	34	23	19	24	16	21	15	24	21	13	22	27	10
$p=\cdot 40$ $q=\cdot 60$	120	118	47	37	33	37	29	35	29	37	35	27	36	41	23
$p=\cdot 35$ $q=\cdot 65$	143	141	70	61	57	60	52	60	54	62	62	53	63	65	46
$p=\cdot 30$ $q=\cdot 70$	180	178	107	100	96	98	89	101	94	100	103	94	105	104	84
$p=\cdot 25$ $q=\cdot 75$	238	235	165	162	156	158	147	165	156	160	168	158	170	166	143
$p=\cdot 20$ $q=\cdot 80$	330	327	259	260	253	253	240	266	257	257	272	260	277	264	238
$p=\cdot 15$ $q=\cdot 85$	491	487	421	430	422	418	402	443	431	425	454	438	461	435	402
$p=\cdot 10$ $q=\cdot 90$	821	816	754	779	768	756	735	807	789	769	825	804	839	785	738

TABLE VI—*Contd.*

Hypothesis	$r=25$	$r=30$		$r=35$	$r=40$		$r=45$			$r=50$	$r=80$		$r=90$	$r=100$	
	W_r'	W_r'	T_r'	W_r'	W_r'	T_r'	W_r	W_r'	T_r	T_r'	W_r'	T_r'	W_r'	W_r or T_r	W_r' or T_r'
$p=\cdot 50$ $q=\cdot 50$	7	6	4	5	5	3	44	4	53	3	3	2	2	138	2
$p=\cdot 45$ $q=\cdot 55$	11	11	8	10	10	7	50	10	58	7	7	6	7	143	6
$p=\cdot 40$ $q=\cdot 60$	26	26	21	26	26	21	68	26	73	20	22	19	20	160	19
$p=\cdot 35$ $q=\cdot 65$	53	54	45	54	55	44	100	56	100	44	48	42	44	191	42
$p=\cdot 30$ $q=\cdot 70$	96	98	83	100	102	83	152	104	142	82	90	79	84	240	78
$p=\cdot 25$ $q=\cdot 75$	163	167	143	171	..	142	232	179	209	141	155	136	..	317	135
$p=\cdot 20$ $q=\cdot 80$	270	278	238	286	293	238	361	298	316	236	259	228	..	440	227
$p=\cdot 15$ $q=\cdot 85$	457	471	403	485	..	404	585	507	502	402	441	388	..	653	386
$p=\cdot 10$ $q=\cdot 90$	840	866	743	892	..	745	1045	934	884	741	814	716	..	1091	713

TABLE VII

$(C.V.)^2$ *of different statistics for* $n = 200$

Hypothesis	$r=2$		$r=10$	$r=20$	$r=30$	$r=40$		$r=60$	$r=80$	$r=200$	
	W_r	W_r'	W_r	W_r	W_r	W_r	W_r'	W_r'	W_r'	W_r	W_r'
$p=\cdot 50$, $q=\cdot 50$	51	50	8	5	5	6	2	1	1	68	0·5
$p=\cdot 45$, $q=\cdot 55$	53	52	10	8	8	9	4	4	4	71	3
$p=\cdot 40$, $q=\cdot 60$	59	59	17	14	15	16	11	12	12	79	9
$p=\cdot 35$, $q=\cdot 65$	71	70	29	27	28	29	25	26	27	94	20
$p=\cdot 30$, $q=\cdot 70$	89	88	48	46	48	51	45	48	50	119	39
$p=\cdot 25$, $q=\cdot 75$	118	117	77	77	80	84	78	83	87	157	67
$p=\cdot 20$, $q=\cdot 80$	164	163	124	126	131	137	130	138	146	218	113
$p=\cdot 15$, $q=\cdot 85$	244	243	207	212	220	229	221	235	249	325	193
$p=\cdot 10$, $q=\cdot 90$	408	407	376	387	402	419	407	434	459	543	356

used, T_n' ($= W_n'$) is far more powerful than W_r or T_r and therefore T_n' should be preferred to W_r and T_r.

6. Summary

The paper deals with the distributions of a number of *statistics* W_r, W_r', T_r, and T_r' defined for a sequence of n random observations from a continuous or discontinuous distribution. In the case of continuous distribution the observations take values from $-\infty$ to $+\infty$ while for the discontinuous case the values taken are $\theta_1 < \theta_2 < \theta_3 \cdots < \theta_k$ with probabilities $p_1, p_2, \cdots p_k$ and include the cases of both free and non-free sampling. The *statistics* W_r refer to total number of positive or negative differences between all possible pairs of observations considered according to the order of occurrence in moving blocks of r contiguous observations. Thus in a sequence of n observations there will be $(n - r + 1)$ blocks each yielding $\binom{r}{2}$ differences. W_r' is composed of the total number of positive and negative differences (excluding the zeroes). T_r is obtained by taking the number of positive or negative differences between pairs of observations, x_i, x_j in the sequence such that $(j - i) \leqslant r - 1$. T_r' includes both positive and negative differences mentioned above for T_r. It may be observed that in the case of W-statistics the difference from any pair of observations will be repeated a number of times on account of overlapping while in T-statistics any of the differences will occur once only. When the distributions are continuous, the statistics W_r' and T_r' are constant for a given sequence irrespective of its order of occurrence for a given r and therefore their distributions do not exist. But all the statistics are definable and useful for testing the homogeneity of two or more samples from continuous distributions. This is done by pooling the samples together and arranging them in ascending or descending order and identifying the observations as 1, 2, 3, etc., according as they belong to samples 1, 2, 3, etc., respectively as in the case of Wald and Wolfowitz's U-statistics.

It has been shown that the distributions of all these *statistics* tend to the normal form as n tends to infinity. The standardized deviates of these statistics can serve as tests for examining (i) the randomness of a given sequence of observations and (ii) the homogeneity of two or more samples.

Detailed examination of the powers and asymptotic relative efficiency (A.R.E.) of these *statistics* shows that whenever W_r' and T_r'

are applicable, T_r' is the most powerful of all the tests for $r = n$. Though W_r and T_r are less powerful and less efficient than W_r' and T_r', in general W_r is more powerful and asymptotically more efficient than T_r. The powers and A.R.E. of W_r and T_r increase with r, attain a maximum and then gradually fall off. The maximum power and A.R.E. attained for W_r and T_r are much more than those for Mann and Whitney's or Wilcoxon's test the A.R.E. of which as compared to t-test is $3/\pi$. The corresponding A.R.E. for W_r or T_r having maximum power appears to be more than unity. In fact, even for the minimum value of r, that is 2, the power and A.R.E. for W_r and T_r are slightly more than those for $r = n$ which corresponds to Wilcoxon's test.

Thus the statistics developed in this paper lead to non-parametric tests which are more powerful than those developed so far. Further investigations are in progress.

Our sincere thanks are due to Shri S. P. Varma for having helped us in the tedious calculations regarding the powers and A.R.E.'s given in this paper.

REFERENCES

Dixon, W. J. .. "Power functions of the sign test and power efficiency for normal alternatives," *Ann. Math. Stat.*, 1953, **24**, 467.

Kendall, M. G. .. *The Advanced Theory of Statistics*, Charles Griffin & Co., Ltd., London, 1945, **1**.

Krishna Iyer, P. V. .. "The theory of probability distributions of points on a line," *J. Ind. Soc. Agri. Stat.*, 1948, **1**, 173.

——— .. "Further contributions to the theory of probability distributions of points on a line," *Ibid.*, 1950, 51 and 52, **2, 3 and 4**, 141, 80 and 50.

——— .. "Some distributions arising in matching problems,' *Ibid.*, 1954, **6**, 5.

——— and Kapur, M. N. "Probability distributions arising from points on a line," *Biometrika*, 1954, **41**, 553.

Kruskal, W. H. .. "A non-parametric test for the several sample problem," *Am. Math. Stat.*, 1952, **23**, 565.

——— and Wallis, W. A. "Use of ranks in one criterion variance analysis," *Jour. Ann. Stat. Assocn.*, 1952, **47**, 583.

Mann, H. B. and Whitney, D. R. "On a test whether one of two random variables is stochastically larger than the other," *Ann. Math. Stat.*, 1947, **18**, 50.

Mood, A. M. .. "The distribution theory of runs," *Ibid.*, 1940, **11**, 367.

Mood, A. M. .. "On the asymptotic efficiency of certain non-parametric two sample tests," *Ibid.*, 1954, **25**, 514.

Pitman, E. J. C. .. "*Lecture Notes on Non-Parametric Statistical Inference*," University of North Carolina, Institute of Statistics (1948).

Rijkoort, P. J. .. "A generalization of Wilcoxon's tests," *Proc. Kon. Ned. Akad. Wetens Chappen* (1952), A 55, 394, 1952.

Singh, B. N. .. "Use of complex Markoff's chain in testing randomness," *J. Ind. Soc. Agri. Stat.*, 1952, **4**, 145.

Stuart, A. .. "Some quick sign tests for trend in location and dispersion," *Biometrika*, 1955, **92**, 80.

Wald, A. .. *Sequential Analysis*, John Willey & Sons, New York, 1945.

—— and Wolfowitz, J. .. "On a test whether two samples are from the same population," *Ann. Math. Stat.*, 1940, **11**, 147.

Walsh, J. E. .. "On the power function of the sign test for slippage of means," *Ibid.*, 1946, **17**, 360.

Whitney, D. R. .. "A bivariate extension of U-Statistics," *Ibid.*, 1951, **22**, 274.

Wilcoxon, F. .. "Individual comparisons by ranking Methods," *Biometrics Bulletin*, 1945, **1**, 80.